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THE EQUATION OF AXISYMMETRIC BUOYANCY OSCILLATIONS IN AN IDEAL FLUID†

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A fourth-order linear partial differential equation is derived to describe axisymmetric oscillations of an ideal incompressible stratified fluid in a gravitational force field. Potential vortices and mass sources are distributed along the axis of symmetry. A class of steady solutions, which depend on three real parameters, is constructed in the linear approximation. The asymptotic behaviour of these solutions at short and long distances from the axis of symmetry is investigated. © 2000 Elsevier Science Ltd. All rights reserved.

1. THE FUNDAMENTAL EQUATION

The system of equations describing the axisymmetric motion of an ideal incompressible stratified fluid in a gravitational force field had the following form [1]

$$\frac{\partial v_r}{\partial t}+\frac{\partial z}{\partial r}D^2 z=-\frac{\partial H}{\partial r},\quad D(rv_\theta)=0 \tag{1.1}$$

$$\left(\frac{N^2}{g}-\frac{\partial}{\partial \zeta}\right)\left(\frac{v_r^2+v_\theta^2}{2}-H\right)+N^2 z+\frac{\partial z}{\partial \zeta}D^2 z=0 \tag{1.2}$$

$$\frac{\partial v_r}{\partial r}+\frac{v_r}{r}+D\ln\left(\frac{\partial z}{\partial \zeta}\right)=0,\quad D=\frac{\partial}{\partial t}+v_r\frac{\partial}{\partial r} \tag{1.3}$$

$$H=\frac{p}{\rho}+\frac{v_r^2+v_\theta^2}{2}+gz,\quad N^2=-\frac{g\rho'(\zeta)}{\rho(\zeta)} \tag{1.4}$$

where r, θ, ζ and t are independent variables: r is the distance of the point from the axis of symmetry, θ is the polar angle, ζ is the distance of a fluid particle in an equilibrium position from a fixed horizontal plane, t is the time, the cartesian coordinate z is the dependent variable, v_r and v_θ are the radial and peripheral velocities of a fluid particle, p is the pressure, g is the acceleration due to gravity, $\rho(z)$ is the density of the particle in an equilibrium position and N is the Brunt–Väisälä frequency.

If the medium is an ideal gas, we will confine ourselves to the range of Brunt–Väisälä frequencies, when acoustic oscillations are insignificant compared with buoyancy oscillations. It has been shown [1] that with this assumption the system of equations again has the form (1.1)–(1.3), but relations (1.4) become

$$H=\frac{a^2}{\varkappa-1}+\frac{v_r^2+v_\theta^2}{2}+gz,\quad N^2=\frac{ga'(\zeta)}{\varkappa a(\zeta)},\quad \varkappa=c_p/c_v$$

where a is the speed of sound and $\ln a(z)$ is the entropy of a gas particle in equilibrium.

The equation of conservation of mass (1.3) is easily written in divergent form

$$\frac{\partial}{\partial r}(rv_r z_\zeta)+\frac{\partial}{\partial t}(rz_\zeta)=0 \tag{1.5}$$

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If we put

$$z=\zeta+w=\zeta+\frac{1}{r}\frac{\partial\omega}{\partial r}$$

$$rv_r\frac{\partial z}{\partial\zeta}=Q(\zeta,\ t)-\frac{\partial^2\omega}{\partial\zeta\partial t},\quad u=\frac{r^2}{2}\tag{1.6}$$

where $Q(\zeta, t)$ is an arbitrary function, Eqs (1.5) will be satisfied and it follows from (1.6) that

$$v_r=\frac{\mathbf{A}\omega}{r},\quad z=\zeta+\frac{\partial\omega}{\partial u},\quad \mathbf{A}\omega=\left(Q-\frac{\partial^2\omega}{\partial t\partial\zeta}\right)\left(1+\frac{\partial^2\omega}{\partial u\partial\zeta}\right)^{-1}\tag{1.7}$$

It follows from the second equation of (1.1) that $v_\theta=\gamma(\zeta)/r$, where the arbitrary function $2\pi\gamma(\zeta)$ is equal to the circulation of the velocity vector around a simple contour in the plane ζ = const.

Eliminating the function H from Eqs (1.1) and (1.2) and using equalities (1.6) and (1.7), we obtain the equation

$$\left(\frac{N^2}{g}-\frac{\partial}{\partial\zeta}\right)\left(\frac{\partial}{\partial u}\left(\frac{\gamma^2+(\mathbf{A}\omega)^2}{4u}\right)+\frac{1}{2u}\frac{\partial(\mathbf{A}\omega)}{\partial t}+\frac{\partial^2\omega}{\partial u^2}\mathbf{D}^2\left(\frac{\partial\omega}{\partial u}\right)\right)+$$

$$+N^2\frac{\partial^2\omega}{\partial u^2}+\frac{\partial}{\partial u}\left(\left(1+\frac{\partial^2\omega}{\partial u\partial\zeta}\right)\mathbf{D}^2\left(\frac{\partial\omega}{\partial u}\right)\right)=0,\quad \mathbf{D}=\frac{\partial}{\partial t}+\mathbf{A}\omega\frac{\partial}{\partial u}$$

2. THE STEADY EQUATIONS

In the steady state, it follows from Eqs (1.5) and (1.1) that

$$v_r=\frac{Q(\zeta)}{r}\left(\frac{\partial z}{\partial\zeta}\right)^{-1},\quad -H(r,\ \zeta)=-g\zeta+\frac{Q^2(\zeta)}{2r^2}\left(\frac{\partial z}{\partial r}\right)^2\left(\frac{\partial z}{\partial\zeta}\right)^{-2}\tag{2.1}$$

where $Q(\zeta)$ is an arbitrary function. If $Q=0$, then

$$z=\zeta-\frac{1}{2N^2r^2}\left(\frac{N^2}{g}-\frac{\partial}{\partial\zeta}\right)\gamma^2(\zeta)$$

In what follows we will confine our attention to the case when

$$\gamma(\zeta)=\text{const},\quad Q(\zeta)=\text{const}\neq 0$$

Substituting expressions (2.1) into Eq. (1.2) and making the change of variables

$$r^2=\frac{2QR}{N},\quad z=\zeta+\alpha W,\quad \alpha=\frac{NQ}{4g}\left(1+\frac{\gamma^2}{Q^2}\right)\tag{2.2}$$

we obtain a non-linear equation

$$\mathbf{L}W+R^{-1}+\mathbf{F}W=0\tag{2.3}$$

$$\mathbf{L}W=\frac{\partial^2W}{\partial R^2}+W+\frac{Q}{2NR}\left(\frac{\partial^2W}{\partial\zeta^2}-\frac{N^2}{g}\frac{\partial W}{\partial\zeta}\right)$$

$$\mathbf{F}W=-\alpha\frac{\partial}{\partial R}\left(\frac{\partial W}{\partial R}\frac{\partial W}{\partial\zeta}\left(1+\alpha\frac{\partial W}{\partial\zeta}\right)^{-1}\right)+$$

$$+\left(\frac{N^2}{g}-\frac{\partial}{\partial\zeta}\right)\left(\frac{Q}{4NR}\left(2\alpha^2\left(\frac{\partial W}{\partial\zeta}\right)^3+3\alpha\left(\frac{\partial W}{\partial\zeta}\right)^2\right)+\frac{\alpha}{2}\left(\frac{\partial W}{\partial R}\right)^2\right)\left(1+\alpha\frac{\partial W}{\partial\zeta}\right)^{-2}$$

3. SOLUTION OF THE NON-LINEAR EQUATION INDEPENDENT OF ζ

If the function W is independent of ζ, Eq. (2.3) becomes

$$\frac{\partial^2 W}{\partial R^2}+W+\frac{1}{R}+\beta\left(\frac{\partial W}{\partial R}\right)^2=0,\quad \beta=\frac{N^3 Q}{8g^2}\left(1+\frac{\gamma^2}{Q^2}\right) \tag{3.1}$$

The solution of Eq. (3.1) as a series in inverse powers of R is

$$W=\sum_{k=1}^{\infty}\frac{c_k}{R^k},\quad c_1=-1,\quad c_2=0,\quad c_3=2$$

$$c_k+\beta\sum_{i=2}^{k}(i-1)(k-i+1)c_{i-1}c_{k-i-1}+(k-1)(k-2)c_{k-2},\quad k\geqslant 4$$

$$c_4=\beta,\quad c_5=-24,\dots$$

4. SOME CLASSES OF SOLUTIONS OF THE LINEAR EQUATION

An exact solution for the non-linear equation (2.3) is difficult to find. Hence forth we will confine ourselves to investigating the linear equation. Dropping the non-linear terms in Eq. (2.3), we obtain

$$\mathbf{L}W+R^{-1}=0 \tag{4.1}$$

A solution of Eq. (4.1), independent of the variable ζ, has the form

$$W_0(R)=\int_R^{\infty}\frac{\sin(R-y)}{y}dy=-\sin R\,\mathrm{Ci}(R)-\cos R\left(\frac{\pi}{2}-\mathrm{Si}(R)\right) \tag{4.2}$$

The following asymptotic formulae hold

$$W_0(R)=-\left(R\ln R+\frac{\pi}{2}+(1-\gamma)R\right)+o(R)\quad\text{as}\quad R\to+0$$

$$W_0(R)=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{2^n n!}{R^{2n+1}}\quad\text{as}\quad R\to+\infty$$

where γ is Euler's constant; the last formula is obtained by integration by parts in (4.2).

The general solution of Eq. (4.1) has the form $W = W_0 + W_1$, where W_1 is a solution of the homogeneous equation $\mathbf{L}W_1 = 0$.

We will seek a class of solutions of this equation in the form

$$W_1=\omega e^{\lambda\zeta},\quad \lambda>0$$

To determine the function ω, we obtain the equation

$$\frac{\partial^2\omega}{\partial R^2}+\omega-\frac{2\eta}{R}\omega=0,\quad \eta=\frac{Q\lambda}{4N}\left(\frac{N^2}{g}-\lambda\right)$$

which is a special case of Coulomb's equation [2]. There is a well-known integral representation of its solutions

$$F_0(\eta,\ R)+iG_0(\eta,\ R)=iC_0^{-1}(\eta)e^{-iR}\int_0^{+\infty} e^{-t}t^{-i\eta}(t+2iR)^{i\eta}dt$$

$$C_0^2(\eta)=2\pi\eta(e^{2\pi\eta}-1)^{-1}$$

containing regular and irregular Coulomb wave functions of zero index.

The regular Coulomb function may be expressed as the sum of a power series

$$F_0(\eta,\ R)=C_0(\eta)\sum_{k=1}^{\infty} A_k(\eta)R^k$$

The coefficients A_k are determined by a recurrence system of equations

$$A_1=1,\ \ A_2=\eta,\ \ k(k-1)A_k=2\eta A_{k-1}-A_{k-2},\ \ k>2$$
$$A_3=-\frac{1}{6}+\frac{\eta^2}{3},\ \ A_4=-\frac{\eta}{9}+\frac{\eta^3}{18},\ \dots \tag{4.3}$$

The irregular Coulomb function has the form

$$G_0(\eta,\ R)=$$
$$=\frac{2\eta}{C_0^2(\eta)}F_0(\eta,\ R)(\ln 2R-1+2\gamma+\operatorname{Re}\psi(1+i\eta))+C_0(\eta)\theta(\eta,\ R)$$

where $\psi(x)$ is the logarithmic derivative of the Gamma function, and the function $\theta(\eta, R)$ may be expanded in a power series

$$\theta(\eta,\ R)=\sum_{k=0}^{\infty} B_kR^k,\ \ B_0=1,\ \ B_1=0$$
$$k(k-1)B_k=2\eta B_{k-1}-B_{k-2}-2\eta(2k-1)A_k,\ \ k>2$$
$$B_2=-\left(\frac{1}{2}+\eta^2\right),\ \ B_3=\left(\frac{\eta}{9}-\frac{14\eta^3}{9}\right),\dots$$

For small R

$$F_0(\eta,\ R)=C_0(\eta)R$$
$$G_0(\eta,\ R)=2\eta RC_0^{-2}(\eta)(\ln 2R-1+2\gamma+\operatorname{Re}\psi(1+i\eta))+C_0(\eta)$$

As $R\to\infty$

$$F_0(\eta,\ R)+iG_0(\eta,\ R)=i\exp(-i(R-\eta\ln 2R-\sigma))\times$$
$$\times\left(1+\sum_{n=1}^{\infty}\left((-1)^n\frac{i\eta(1-i\eta)^2\dots(n-1-i\eta)^2(n-i\eta)}{(2iR)^n n!}\right)\right),\ \ \sigma=\arg\Gamma(1+i\eta)$$

Confining ourselves to the principal terms of the asymptotic series, we obtain

$$G_0(\eta,\ R)=\cos(R-\eta\ln 2R-\sigma),\ \ F_0(\eta,\ R)=\sin(R-2\eta\ln R-\sigma)$$

It follows from relations (2.2) that we have constructed a class of solutions of the form

$$z=\zeta+\alpha(W_0(R)+C_1(\eta)\cos R+C_2(\eta)\sin R)+$$
$$+e^{\lambda\zeta}(C_3(\eta)F_0(\eta,\ R)+C_4(\eta)G_0(\eta,\ R))$$

where $C_k(\eta)$ are arbitrary functions, the function $W_0(R)$ is defined by (4.2), and $F_0(\eta, R)$ and $G_0(\eta, R)$ are Coulomb wave functions of zero index.

We can similarly construct another class of solutions

$$z = \zeta + W_0(R) + c_1(\lambda)\sin R + c_2(\lambda)\cos R + \alpha \exp\left(\frac{N^2}{2g} + i\lambda\sqrt{\frac{2N}{Q}}\right)\zeta \times$$

$$\times (c_3(\lambda) F_0(\delta(\lambda),\ R) + c_4(\lambda) G_0(\delta(\lambda),\ R)),\quad \delta(\lambda) = \frac{N^3 Q}{16 g^2} - \frac{\lambda^2}{2}$$

More general classes of solutions may be obtained by integrating this expression with respect to the parameter λ.

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